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# Variation on Butzer's Problem: Characterization of the Solutions

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Dedicated to Professor P. L. Butzer

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**Abstract**—The so-called Butzer problem had been a challenge to the approximation community for decades until its first constructive solution was given by Cao and Gonska at the end of the 1980s. Butzer's original problem can be turned into a "strong form", namely a pointwise one. In the present note, we characterize, among other things, and in an easy way, certain operators which solve Butzer's problem in this strong and most demanding version.

**Keywords**—Butzer's problem, Discretely defined positive linear operators, DeVore-Gopengauz inequality, Piecewise linear interpolation, Optimal positive linear operators.

## 1. INTRODUCTION AND HISTORICAL NOTES

At the 1980 Budapest conference, Butzer [1] posed the following problem, which he said "... has probably been part of the mathematical folklore for some generations ..." (see also [2]).

PROBLEM 1. BUTZER'S PROBLEM.

Can one construct a triangular matrix of distinct nodes

$$\{x_{k,n}\}_{k=0}^{n}, \qquad (n \in \mathbb{N}_0,\ 0 \le x_{k,n} \le 1),$$

and a triangular matrix of positive fundamental functions $\{\phi_{k,n}\}_{k=0}^{n}$ defined on $I = [0,1]$, such that the linear summator operators

$$L_n(f,x) := \sum_{k=0}^{n} f(x_{k,n})\,\phi_{k,n}(x), \qquad f \in C(I), \tag{1}$$

give algebraic polynomials of degree $n$, and satisfy

$$\|L_n f - f\|_{C(I)} = \mathcal{O}\left(n^{-\alpha}\right), \tag{2}$$

provided $f \in \mathrm{Lip}_2(\alpha, C)$, $0 < \alpha \le 2$, i.e., $\omega_2(f,\delta)_{C(I)} \le C \cdot \delta^{\alpha}$?

Recall that under the same hypotheses, the Bernstein polynomials approximate $f$ only with order $\mathcal{O}(n^{-\alpha/2})$.

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Another related problem that can be raised is the following problem.

PROBLEM 2.

Given a triangular matrix of distinct nodes

$$\{x_{k,n}\}_{k=0}^{n}, \qquad (n \in \mathbb{N}_0,\ 0 \le x_{k,n} \le 1),$$

can one construct a triangular matrix of positive fundamental functions $\{\phi_{k,n}\}_{k=0}^{n}$ defined on $I$, such that the linear summator operators

$$L_n(f,x) := \sum_{k=0}^{n} f(x_{k,n})\phi_{k,n}(x), \qquad f \in C(I),$$

give algebraic polynomials of degree $n$, and satisfy

$$\|L_n f - f\|_{C(I)} = \mathcal{O}(n^{-\alpha}),$$

provided $f \in \mathrm{Lip}_2(\alpha, C)$, $0 < \alpha \le 2$?

The first solution to Butzer's problem (positive linear form) was given by Cao and Gonska in 1989 [3]; others followed in [4,5]. Later, other authors also gave solutions to Butzer's problem using methods similar to those in [3]; see, e.g., [6,7]. Furthermore, in [4], Cao and Gonska also constructed, via the Boolean sum approach, linear polynomial operators satisfying DeVore-Gopengauz inequalities, namely

$$|L_n(f;x) - f(x)| \le c\,\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \tag{3}$$

for all $f \in C(I)$ and all $x \in I$, with the constant $c$ independent of $f$, $n$, 3, and $x$.

Since the Boolean sum of two positive linear operators is not a positive linear operator in general, Gonska and Zhou in [8] formulated the following problem.

PROBLEM 3.

Do there exist positive linear operators $L_n : C(I) \to \Pi_n$ such that for all $f \in C(I)$ and all $x \in I$, one has

$$|L_n(f;x) - f(x)| \le c\,\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right),$$

with the constant $c$ independent of $f, n$, and $x$?

Recently Gavrea [9] constructed *nondiscrete* positive linear operators satisfying (3); thus, he completely solved Problem 3.

Combining Problems 1 and 3 leads to the question whether there exist *discretely defined* positive linear operators satisfying (3). We will call this new problem **the strong form of Butzer's problem**. In [10], we gave the first solution to it, and we generalized it in [11]. In the latter paper, we also constructed for the first time positive linear operators with equidistant nodes solving Butzer's problem (Problem 1). Moreover, in [12, Theorem 5], we constructed operators which provide the first solution to Butzer's problem for positive linear operators with equidistant nodes, and satisfying estimates in terms of $\omega_2$ only.

In the present paper, we will characterize the solutions to Problems 1, 2, and to the "strong form of Butzer's problem". We mention here that $c$ will always denote a numerical constant independent of other quantities in question, unless otherwise indicated.

## 2. CHARACTERIZATION OF THE SOLUTIONS TO PROBLEM 1

THEOREM 1. *Let $L_n : C(I) \to \Pi_n$ be a sequence of positive linear operators of the form (1). The operators $L_n$ verify Butzer's problem if and only if they satisfy the conditions*

$$\|L_n e_i - e_i\| \le \frac{c}{n^2}, \qquad i = 0, 1, 2, \tag{4}$$

*with the positive constant c independent of n.*

PROOF. In order to prove the sufficiency, we use the following result established by Gonska and Kovacheva in [13].

THEOREM 2. *If $L : C[a,b] \to C[a,b]$ is a positive linear operator, then for $f \in C[a,b]$, $x \in [a,b]$, and each $0 < h \le 1/2(b-a)$, the following holds:*

$$\begin{aligned} |(Lf)(x) - f(x)| \le{}& |(Le_0)(x) - 1| \cdot \|f\| + \frac{2}{h} \cdot |(L(e_1 - x))(x)| \cdot \omega_1(f;h) \\ &+ \left[\frac{3((Le_0)(x)+1)}{4} + \frac{3}{4}|(Le_0)(x) - 1| + \frac{3}{2h} \cdot |(L(e_1 - x))(x)| \right. \\ &\left. + \frac{3}{4h^2} \cdot \left(L(e_1 - x)^2\right)(x)\right] \cdot \omega_2(f;h). \end{aligned} \tag{5}$$

*From (5), we immediately obtain*

$$\begin{aligned} \|L_n f - f\| \le{}& \|L_n e_0 - e_0\| \cdot \|f\| + \frac{2}{h} \cdot \|L_n e_1 - e_1 \cdot L_n e_0\| \cdot \omega_1(f;h) \\ &+ \left[\frac{3}{4}(L_n e_0 + 1) + \frac{3}{4}\|L_n e_0 - 1\| + \frac{3}{2h} \cdot \|L_n e_1 - e_1 \cdot L_n e_0\| \right. \\ &\left. + \frac{3}{4h^2} \cdot \|L_n e_2 - 2e_1 \cdot L_n e_1 + e_2 \cdot L_n e_0\|\right] \cdot \omega_2(f;h). \end{aligned} \tag{6}$$

*From conditions (4), we obtain that*

$$\begin{aligned} &\|L_n e_1 - e_l \cdot L_n e_0\| = \|L_n e_1 - e_1 + e_1(1 - L_n e_0)\| \le \|L_n e_1 - e_1\| + \|L_n e_0 - 1\| \le \frac{c}{n^2}, \\ &\quad L_n e_0 + 1 \le 2 + \frac{c}{n^2}, \end{aligned}$$

*and*

$$\begin{aligned} \|L_n e_2 - 2e_1 \cdot L_n e_1 + e_2 \cdot L_n e_0\| &= \|L_n e_2 - e_2 + e_2(L_n e_0 - 1) + 2e_1(e_1 - L_n e_1)\| \\ &\le \|L_n e_2 - e_2\| + \|L_n e_0 - 1\| + 2\|L_n e_1 - e1\| \le \frac{c}{n^2}. \end{aligned}$$

*Choosing $h = 1/n$ for $n \ge 1$ in (6), and taking into account the latter inequalities, yields*

$$\|L_n f - f\| \le \frac{c}{n^2} \cdot \|f\| + \frac{c}{n} \cdot \omega_1\left(f; \frac{1}{n}\right) + c \cdot \omega_2\left(f; \frac{1}{n}\right).$$

*The previous inequality implies that, for $f \in \mathrm{Lip}_2(\alpha, C)$, $0 < \alpha \le 2$, one has*

$$\|L_n f - f\|_{C(I)} = \mathcal{O}\left(n^{-\alpha}\right),$$

*which means that $L_n$ provides a solution to Butzer's problem.*

*In order to prove the necessity, we assume that $L_n$ gives a solution to Butzer's problem, and we show that $L_n$ satisfies conditions (4). It is trivial that $e_i \in \mathrm{Lip}_2(2, C)$, $i = 0, 1, 2$. Since $L_n$ verifies Butzer's problem, it follows*

$$\|L_n e_i - e_i\| \le \frac{c}{n^2}. \tag{7}$$

*This completes the proof of Theorem 4.* ∎

## 3. ON PROBLEM 2

THEOREM 3. *Let $L_n : C(I) \to \Pi_n$ be a sequence of positive linear operators of the form (1), which give a positive answer to Butzer's problem. Let the partition $\Delta_n$ be given by*

$$\Delta_n : 0 \le x_{0,n} < x_{1,n} < \cdots < x_{n,n} \le 1,$$

*and let its norm be defined by*

$$\|\Delta_n\| := \max_{k \in \{0,\dots,n-1\}} (x_{k+1,n} - x_{k,n}).$$

*Then there exists a constant c independent of n, such that*

$$\|\Delta_n\| \le \frac{c}{n}.$$

PROOF. Let $\|\Delta_n\| = x_{i+1,n} - x_{i,n}$. We consider the function $f := (e_1 - (x_{i,n} + x_{i+1,n}/2))^2 \in \mathrm{Lip}_2(2, C)$. It follows

$$\|L_n f - f\| \le \frac{c}{n^2}.$$

But

$$\|L_n f - f\| \ge (L_n f)\left(\frac{x_{i,n} + x_{i+1,n}}{2}\right).$$

We have

$$(L_n f)\left(\frac{x_{i,n} + x_{i+1,n}}{2}\right) = \sum_{k=0}^{n} \phi_{k,n}\left(\frac{x_{i,n} + x_{i+1,n}}{2}\right) \cdot \left(x_{k,n} - \frac{x_{i,n} + x_{i+1,n}}{2}\right)^2.$$

Since

$$\left(x_{k,n} - \frac{x_{i,n} + x_{i+1,n}}{2}\right)^2 \ge \frac{\|\Delta_n\|^2}{4},$$

it follows

$$(L_n f)\left(\frac{x_{i,n} + x_{i+1,n}}{2}\right) \ge \frac{\|\Delta_n\|^2}{4}(L_n e_0)\left(\frac{x_{i,n} + x_{i+1,n}}{2}\right).$$

Thus,

$$\frac{\|\Delta_n\|^2}{4}(L_n e_0)\left(\frac{x_{i,n} + x_{i+1,n}}{2}\right) \le \frac{c}{n^2},$$

implying $\|\Delta_n\|^2/4 \le c/n^2$, which then gives $\|\Delta_n\| \le c/n$. ∎

Our next theorem will give necessary and sufficient conditions for solving Problem 2 subject to the additional assumption that all the partitions $\Delta_n$ include the endpoints. To that end, we will consider $f \in C[0,1]$, a partition $\Delta_n$ of the interval $[0,1]$,

$$\Delta_n : 0 = x_{0,n} < x_{1,n} < \cdots < x_{n,n} = 1,$$

and we will denote by $S_{\Delta_n} f$, the piecewise linear function that interpolates $f$ at the points $x_{k,n}$, $k = 0, \dots, n$.

REMARK 4.

(i) Several representations for the operator $S_{\Delta_n}$ are known. For the sake of completeness, we mention here the following representation given by Popoviciu [14, p. 151] (although we will not use any particular representation of $S_{\Delta_n}$). For every function $f$ defined at the points $x_{k,n}$, $k = 0, \dots, n$, there holds

$$(S_{\Delta_n} f)(x) = f(x_{0,n}) + x\,[x_{0,n}, x_{1,n}; f] + \sum_{k=2}^{n} \frac{x_{k,n} - x_{k-2,n}}{2}\left(|x - x_{k-1,n}| + x - x_{k-1,n}\right) \cdot [x_{k-2,n}, x_{k-1,n}, x_{k,n}; f].$$

(ii) It is known that the operator $S_{\Delta_n}$ preserves linear functions.

(iii) $S_{\Delta_n}$ is a positive linear operator if and only if $x_{0,n} = 0$ and $x_{n,n} = 1$. In case one of these conditions is not satisfied, it is an easy matter to give an example of a positive function $f$, for which $S_{\Delta_n}(f;0)$ and/or $S_{\Delta_n}(f;1)$ is strictly negative.

THEOREM 5. *Let $L_n : C(I) \to \Pi_n$ be a sequence of positive linear operators. Then the operators $\mathcal{L}_n : C(I) \to \Pi_n$, defined by $\mathcal{L}_n := L_n \circ S_{\Delta_n}$, will verify Butzer's problem if and only if the following conditions hold:*

(i) $\|L_n e_i - e_i\| \le c/n^2$, $i = 0,1,2$,

(ii) $\|\Delta_n\| \le c/n$, *with the positive constant $c$ independent of $n$.*

PROOF. For sufficiency we use, as in the proof of Theorem 1, the inequality (5) by Gonska and Kovacheva. It is enough then to notice that

$$\mathcal{L}_n\left((e_1 - x)^2; x\right) = L_n\left(S_{\Delta_n}(e_1 - u)^2; u\right)(x) + L_n\left((e_1 - x)^2; x\right),$$
$$\left|S_{\Delta_n}\left((e_1 - u)^2; u\right)\right| \le \frac{\|\Delta_n\|^2}{4},$$

and

$$\mathcal{L}_n e_i = L_n e_i, \qquad i = 0,1.$$

In order to show the necessity, we assume that $\mathcal{L}_n$ solves Butzer's problem. Then from Theorem 3, it follows $\|\Delta_n\| \le c/n$. Since $\mathcal{L}_n e_i = L_n e_i$, $i = 0,1$, from Theorem 1, we obtain

$$\|L_n e_i - e_i\| \le \frac{c}{n^2}, \qquad i = 0,1.$$

We have

$$\begin{aligned}\|L_n e_2 - e_2\| &= \|L_n(e_2 - S_{\Delta_n} e_2) + \mathcal{L}_n e_2 - e_2\| \\ &\le \|L_n e_0\| \frac{\|\Delta_n\|^2}{4} + \|\mathcal{L}_n e_2 - e_2\| \\ &\le \frac{c}{n^2} + \|\mathcal{L}_n e_2 - e_2\|.\end{aligned}$$

But $\mathcal{L}_n$ verifies Butzer's problem, so from Theorem 1, it follows

$$\|\mathcal{L}_n e_2 - e_2\| \le \frac{c}{n^2}, \qquad \text{implying } \|L_n e_2 - e_2\| \le \frac{c}{n^2}.$$

This completes the proof of the theorem. ∎

REMARK 6. For solving Problem 2 (subject to the additional assumptions $x_{0,n} = 0$ and $x_{1,n} = 1$), it is necessary and sufficient to have the triangular matrix $\{x_{k,n}\}_{k=0}^{n}$ given such that $\|\Delta_n\| \le c/n$, and *any* sequence of positive linear operators satisfying (4).

As examples of positive linear operators satisfying (4), we mention here the algebraic polynomial operators $G_{m(n)} : C[-1,1] \to \Pi_{m(n)}$ investigated by, among others, Cao and Gonska in [4] and a series of related papers.

$$\left(G_{m(n)} f\right)(x) = \int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} K_{m(n)}(t,x)\,dt,$$

where $K_{m(n)}(t,x) : [-1,1] \times [-1,1] \to \mathbb{R}_+$ is a positive kernel of the form

$$K_{m(n)}(t,x) = \frac{2}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{m(n)} \rho_{k,m(n)} T_k(t) T_k(x)\right),$$

with $m(n) \in \mathbb{N}$ such that there exist positive constants $c_1$ and $c_2$ independent of $n$, with $c_1 n \le m(n) \le c_2 n$. The $\rho_{k,m(n)}$, $k = 1, \ldots, m(n)$, are the convergence factors and $T_k(x) = \cos(k \arccos x)$ is the $k^{\text{th}}$ Čebyšev polynomial of the $1^{\text{st}}$ kind. We mention that the kernel $K_{m(n)}(t,x)$ is normalized by the condition

$$\int_{-1}^{1} \frac{K_{m(n)}(t,x)}{\sqrt{1-t^2}}\, dt = 1.$$

The operator $G_{m(n)}$ relative to the interval $[0,1]$ is written as follows:

$$\left(G_{m(n)} f\right)(x) = \int_0^1 \frac{f(t) \cdot K_{m(n)}(2t-1, 2x-1)}{\sqrt{t(1-t)}}\, dt. \tag{8}$$

Gonska and Cao [15] proved the following results:

$$\begin{aligned} &G_{m(n)} e_0 = e_0, \\ &\left(G_{m(n)}(e_1 - x)\right)(x) = \frac{1-2x}{2}\left(1 - \rho_{1,m(n)}\right), \\ &\left(G_{m(n)}(e_1 - x)^2\right)(x) = \frac{x(1-x)}{2}\left(1 - \rho_{2,m(n)}\right) + \frac{(2x-1)^2}{4}\left\{\frac{3}{2} - 2\rho_{1,m(n)} + \frac{1}{2}\rho_{2,m(n)}\right\}. \end{aligned} \tag{9}$$

We will present kernels for which $1 - \rho_{1,m(n)} \le c/n^2$. Then (see [15]) also $1 - \rho_{2,m(n)} \le c/n^2$, with the constant $c$ independent of $n$.

Thus, relations (9) imply that

$$\left|\left(G_{m(n)}(e_1 - x)\right)(x)\right| \le \frac{c}{n^2},$$

and

$$\left(G_{m(n)}(e_1 - x)^2\right)(x) \le \frac{c}{n^2},$$

so conditions (4) are satisfied.

Kernels $K_{m(n)}$ for which the corresponding operators $G_{m(n)}$ have $1 - \rho_{1,m(n)} \le c/n^2$ are, for example,

- Jackson-Matsuoka kernels of order $s \ge 2$ (see [16]),

$$\frac{\pi}{2} \cdot K_{sn-s}(t,x) = \frac{1}{A_{s,s}(n)} \cdot \left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s},$$

  where $v := v(t,x) := \arccos t - \arccos x$, $t, x \in [-1,1]$, and $A_{s,s}(n)$ is a normalizing constant,

- Fejér-Korovkin kernel (see [17]),

$$\frac{\pi}{2} \cdot K_n(t,x) = \frac{1}{n+2}\left(\frac{\sin(\pi/n+2) \cdot \cos(n+2)v/2}{\cos v - \cos(\pi/n+2)}\right)^2,$$

  where $v$ is as above, $v := v(t,x) = \arccos t - \arccos x$,

- Jackson-de La Vallée Poussin kernel (see [5]),

$$\frac{\pi}{2} \cdot K_{2n-1}(t,x) = \frac{2 + \cos v}{4n^3} \cdot \left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^4,$$

  with $v$ as above.

## 4. THE STRONG FORM OF BUTZER'S PROBLEM

THEOREM 7. *Let $L_n : C(I) \to \Pi_n$ be a sequence of positive linear operators of the form (1), which solve the strong form of Butzer's problem. Then one has:*

(i) $L_n e_i = e_i$, $i = 0, 1$,
(ii) $(L_n e_2)(x) - x^2 \le 2c\,(x(1-x))/n^2$,
(iii) $x_{0,n} = 0$, $x_{n,n} = 1$,
(iv) $(x_{k+1,n} - x_{k,n})^2 \le 2c\,((x_{k+1,n} + x_{k,n})(2 - x_{k+1,n} - x_{k,n}))/n^2$, $k = 0, \dots, n-1$.

PROOF. Conditions (i) and (ii) are satisfied, since $L_n$ solve the strong form of Butzer's problem, i.e., (see (3))

$$|L_n(f;x) - f(x)| \le c\,\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right),$$

for all $f \in C(I)$ and all $x \in I$, with the constant $c$ independent of $f$, $n$, and $x$.

We mention here that constant $c$ in the latter inequality is the same with the one appearing in conditions (ii) and (iv) of the theorem.

From the latter inequality, it also follows that the operators $L_n$ interpolate at the endpoints, that is,

$$(L_n f)(0) = f(0) \quad \text{and} \quad (L_n f)(1) = f(1).$$

Applying these equalities for the functions

$$f_0(x) = (x - x_{1,n})^2 \cdot \ldots \cdot (x - x_{n,n})^2,$$

and

$$f_n(x) = (x - x_{0,n})^2 \cdot \ldots \cdot (x - x_{n-1,n})^2,$$

yields the inequalities

$$\begin{aligned} x_{1,n}^2 \cdot \ldots \cdot x_{n,n}^2 &= (x_{0,n} - x_{1,n})^2 \cdot \ldots \cdot (x_{0,n} - x_{n,n})^2 \cdot \Phi_{n,0}(0) \\ &\le (x_{0,n} - x_{1,n})^2 \cdot \ldots \cdot (x_{0,n} - x_{n,n})^2, \end{aligned}$$

and

$$\begin{aligned} (1 - x_{0,n})^2 \cdot \ldots \cdot (1 - x_{n-1,n})^2 &= (x_{n,n} - x_{0,n})^2 \cdot \ldots \cdot (x_{n,n} - x_{n-1,n})^2 \cdot \Phi_{n,n}(1) \\ &\le (x_{n,n} - x_{0,n})^2 \cdot \ldots \cdot (x_{n,n} - x_{n-1,n})^2. \end{aligned}$$

On the other hand, it is trivial that

$$x_{1,n}^2 \cdot \ldots \cdot x_{n,n}^2 \ge (x_{0,n} - x_{1,n})^2 \cdot \ldots \cdot (x_{0,n} - x_{n,n})^2,$$

and

$$(1 - x_{0,n})^2 \cdot \ldots \cdot (1 - x_{n-1,n})^2 \ge (x_{n,n} - x_{0,n})^2 \cdot \ldots \cdot (x_{n,n} - x_{n-1,n})^2.$$

So the above relations imply $x_{0,n} = 0$, $x_{n,n} = 1$.

Let $k \in \{0, \dots, n-1\}$ be fixed, and consider the function $g_k(x) := (x - (x_{k,n} + x_{k+1,n})/2)^2 \in \mathrm{Lip}_2(2, C)$. We have

$$\sum_{i=0}^{n} \Phi_{n,i}(x) \cdot \left(x_i - \frac{x_{k,n} + x_{k+1,n}}{2}\right)^2 - \left(x - \frac{x_{k,n} + x_{k+1,n}}{2}\right)^2 \le 2c\,\frac{x(1-x)}{n^2},$$

for every $x \in [0, 1]$.

Taking in the latter inequality $x = (x_{k,n} + x_{k+1,n})/2$, and taking into account the fact that $(x_i - (x_{k,n} + x_{k+1,n})/2)^2 \geq ((x_{k+1,n} - x_{k,n})^2)/4$, we obtain that condition (iv) is verified. ∎

REMARK 8. Taking $k = 0$ and $k = n - 1$ in inequality (iv) from Theorem 7, we obtain the following upper bounds, respectively:

$$x_{1,n} \leq \frac{4c}{n^2 + 2c},$$
$$1 - x_{n-1,n} \leq \frac{4c}{n^2 + 2c},$$

and $x_{k+1,n} - x_{k,n} \leq c_1/n$, $k = 1, \dots, n-2$, where $c_1$ is a positive constant independent of $n$.

REMARK 9. In [11], we constructed certain operators $\mathcal{H}^*_{n+s+2}$, $s \geq 1$ fixed, which solve the strong form of Butzer's problem. These operators were obtained using arbitrary quadrature formulae of the form $\int_0^1 f(x)\,dx = \sum_{k=1}^{n+s} A_{k,n+s} \cdot f(x_{k,n+s}) + R(f)$ with positive coefficients $A_{k,n+s}$ and degrees of exactness $\geq n + s + 2$. From Remark 8, it follows now that, for any sequence of quadrature formulae of the type just described, one has

$$x_{1,n+s} \leq \frac{c_2}{n^2} \quad \text{and} \quad 1 - x_{n+s,n+s} \leq \frac{c_2}{n^2},$$

where $c_2$ is a constant independent of $n$, and the distance between two consecutive nodes $\leq c_1/n$, with $c_1$ independent of $n$.

THEOREM 10. *Let $L_n : C(I) \to \Pi_n$, $n \in \mathbb{N}$ be a sequence of positive linear operators which preserve linear functions and which satisfy condition (ii) from Theorem 7. Consider partitions of the interval $[0,1]$ of the form*

$$\Delta_n : 0 = x_{0,n} < x_{1,n} < \cdots < x_{n,n} = 1$$

*(so that condition (iii) from Theorem 7 is fulfilled). The operators $\mathcal{L}_n := L_n \circ S_{\Delta_n}$ solve the strong form of Butzer's problem if and only if there exists a constant $A$ independent of $n$ and $k$, such that*

$$(x_{k+1,n} - x_{k,n})^2 \leq 2A\,\frac{(x_{k+1,n} + x_{k,n})\,(2 - x_{k+1,n} - x_{k,n})}{n^2} \tag{10}$$

*holds for $k = 0, \dots, n-1$.*

PROOF. From Theorem 7, it is immediate that condition (10) is necessary.

Assume now that the nodes of $\Delta_n$ satisfy condition (10). An easy computation yields

$$(x - x_{k,n})\,(x_{k+1,n} - x) \leq 2A\,\frac{x(1-x)}{n^2} + \frac{B}{n^4},$$

for every $x \in [0,1]$, $n \geq 1$, where the constant $B = \max(2A^2, [2\sqrt{A}]^4)$.

With $A$ and $B$ as above, one has

$$\begin{aligned}
\mathcal{L}_n\left((t-x)^2; x\right) &= L_n\left(S_{\Delta_n}(e_2; t) - t^2\right)(x) + L_n\left((t-x)^2; x\right) \\
&\leq \frac{2A}{n^2}\, L_n(t(1-t); x) + \frac{B}{n^4} + \frac{c}{n^2}\, x(1-x) \\
&\leq \frac{2A}{n^2}\, x(1-x) + \frac{B}{n^4} + \frac{c}{n^2}\, x(1-x) \\
&\leq \frac{c + 2A}{n^2}\, x(1-x) + \frac{B}{n^4}.
\end{aligned}$$

Here in the next to the last step, we used Jensen's inequality for concave functions. It follows then that

$$\mathcal{L}_n\left((t-x)^2; x\right) = \mathcal{O}\left(\frac{x(1-x)}{n^2} + \frac{1}{n^4}\right),$$

where $\mathcal{O}$ is the Landau symbol. Before continuing the proof, we recall the following result given by Cao and Gonska [18].

LEMMA 11. *Let $n \geq 2$, $m(n) \in \mathbb{N}$, and $cn \leq m(n) \leq \tilde{c}n$. Furthermore, let $A_n : C[0,1] \to \Pi_{m(n)}$ be a sequence of positive linear operators, satisfying the following conditions:*

(i) $A_n(1,x) = 1$,
(ii) $A_n(t,x) = \lambda_n x$, $1 - \lambda_n = \mathcal{O}(n^{-2})$,
(iii) $A_n((t-x)^2, x) = \mathcal{O}((x(1-x)/n^2) + 1/n^4)$.

*Then we have for $f \in C[0,1]$,*

$$\left|A_n^+(f,x) - f(x)\right| \leq c\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \qquad x \in [0,1],$$

*where $A_n^+ := L \oplus A_n$, and $L$ denotes the first degree Lagrange interpolator at 0 and 1.*

Notice that the operators $\mathcal{L}_n$ satisfy all the requirements of Lemma 11, so for every $f \in C[0,1]$,

$$\left|\mathcal{L}_n^+(f,x) - f(x)\right| \leq c\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \qquad x \in [0,1]$$

holds. Observing that the operators $L_n$ interpolate at 0 and 1, one has $\mathcal{L}_n^+ f = \mathcal{L}_n f$ for every $f \in C[0,1]$, which means that $\mathcal{L}_n$ also satisfy the DeVore-Gopengauz inequality

$$\left|\mathcal{L}_n(f,x) - f(x)\right| \leq c\omega_2\left(f; \frac{\sqrt{x(1-x)}}{n}\right), \qquad x \in [0,1],$$

and thus, the operators $\mathcal{L}_n$ provide a solution for the strong form of Butzer's problem. ∎

THEOREM 12. *Let $\Delta_n$, $n \in \mathbb{N}$ be a sequence of partitions of the interval $[0,1]$ of the form*

$$\Delta_n : 0 = x_{0,n} < x_{1,n} < \cdots < x_{n,n} = 1.$$

*The necessary and sufficient condition in order to exist positive linear operators $L_n$ of the form (1) which solve the strong form of Butzer's problem is to exist a sequence of partitions of the interval $[0, (\pi/2)]$,*

$$\delta_n : 0 = \theta_{0,n} < \theta_{1,n} < \cdots < \theta_{n,n} = \frac{\pi}{2},$$

*such that*

(i) $x_{k,n} = \sin^2 \theta_{k,n}$, $k = 0, \dots, n$,
(ii) $\theta_{k+1,n} - \theta_{k,n} \leq c/n$, $k = 0, \dots, n-1$,

*where $c$ is a constant independent of $n$ and $k$.*

PROOF. It is obvious that for every $x \in [0,1]$, there exists $\theta \in [0,(\pi/2)]$, such that $x = \sin^2\theta$. It follows that there exist partitions $\delta_n$ such that $x_{k,n} = \sin^2\theta_{k,n}$, $k = 0, \dots, n$.

For the sufficiency, we will prove that the nodes $x_{k,n}$ from our theorem satisfy the requirements of Theorem 10.

We write

$$E_{k,n} := \frac{(x_{k+1,n} - x_{k,n})^2}{(x_{k+1,n} + x_{k,n})(2 - x_{k+1,n} - x_{k,n})}.$$

Then,

$$E_{k,n} = \frac{\sin^2(\theta_{k+1,n} - \theta_{k,n}) \cdot \sin^2(\theta_{k+1,n} + \theta_{k,n})}{1 - \cos^2(\theta_{k+1,n} - \theta_{k,n}) \cdot \cos^2(\theta_{k+1,n} + \theta_{k,n})}.$$

Since

$$\sin^2(\theta_{k+1,n}+\theta_{k,n}) \le 1-\cos^2(\theta_{k+1,n}-\theta_{k,n})\cdot\cos^2(\theta_{k+1,n}+\theta_{k,n}),$$

we obtain

$$E_{k,n} \le \sin^2(\theta_{k+1,n}-\theta_{k,n}) \le \frac{c^2}{n^2}.$$

Thus, the requirements (10) of Theorem 10 are satisfied ($A = c^2/2$).

Consider now a sequence of positive linear operators $L_n : C(I) \to \Pi_n$ which satisfy conditions (i) and (ii) from Theorem 7. Then, from Theorem 10, it follows that the operators $\mathcal{L}_n := L_n \circ S_{\Delta_n}$ are of the form (1) and solve the strong form of Butzer's problem. (For concrete examples of $L_n$, see Remark 13.)

We prove now the necessity of our conditions. To that end, we consider a sequence of operators $H_n : C(I) \to \Pi_n$ of the form (1) which verify the strong form of Butzer's problem. From Theorem 7, it follows that $x_{0,n} = 0$, $x_{n,n} = 1$. There exist then a a sequence of partitions

$$\delta_n : 0 = \theta_{0,n} < \theta_{1,n} < \cdots < \theta_{n,n} = \frac{\pi}{2},$$

such that $x_{k,n} = \sin^2\theta_{k,n}$, $k = 0,\dots,n$. Since the operators $H_n$ solve the strong form of Butzer's problem, from Theorem 10, it follows that there is a constant $A$ such that

$$E_{k,n} \le \frac{2A}{n^2}.$$

Because

$$1-\cos^2(\theta_{k+1,n}-\theta_{k,n})\cdot\cos^2(\theta_{k+1,n}+\theta_{k,n}) \le 1-\cos^4(\theta_{k+1,n}+\theta_{k,n}), \qquad k = 0,\dots,n-1,$$

we obtain

$$E_{k,n} \ge \frac{\sin^2(\theta_{k+1,n}-\theta_{k,n})\cdot\sin^2(\theta_{k+1,n}+\theta_{k,n})}{(1+\cos^2(\theta_{k+1,n}+\theta_{k,n}))\cdot\sin^2(\theta_{k+1,n}+\theta_{k,n})} \ge \frac{\sin^2(\theta_{k+1,n}-\theta_{k,n})}{2}.$$

From the previous inequalities, we get

$$\sin^2(\theta_{k+1,n}-\theta_{k,n}) \le \frac{4A}{n^2},$$

implying

$$\theta_{k+1,n}-\theta_{k,n} \le \frac{\pi\sqrt{A}}{n}.$$

Thus, the condition in the theorem is satisfied ($c = \pi\sqrt{A}$). ∎

REMARK 13. Positive linear operators $L_n : C(I) \to \Pi_n$ satisfying conditions (i) and (ii) from Theorem 7 (thus providing also solutions to Problem 3) were constructed by Gavrea in [9] and the authors in [10,11].

It is known that it is impossible to construct a sequence of operators of the form (1) with equidistant nodes which verify the strong form of Butzer's problem (see [18,19]). Our next theorem shows that for every $\delta \in (0,(1/2))$ fixed, there exists a sequence of summator operators, such that for every $n > N_0$, the nodes of the operators in the interval $[\delta, 1-\delta]$ are equidistant, and the operators solve the strong form of Butzer's problem.

THEOREM 14. *Let $\delta \in (0,(1/2))$ be fixed. There exists a sequence of summator operators of the form (1) and a fixed natural number $N_0$ such that the following hold.*

(i) *For $n \le N_0$, the nodes of $\Delta_n$ are equidistant.*
(ii) *For $n > N_0$, the nodes in the range $[\delta, 1-\delta]$ are equidistant.*
(iii) *The sequence solves the strong form of Butzer's problem.*

PROOF. Let $N_0 := [1/\delta] + 1$. We may disregard the cases $n \le N_0$, since in these we have the Bernstein operators giving

$$\begin{aligned}|B_n(f,x) - f(x)| &\le c\cdot\omega_2\left(f;\frac{\sqrt{x(1-x)}}{\sqrt{n}}\right)\\ &\le c\cdot\left(\sqrt{n}+1\right)^2\cdot\omega_2\left(f;\frac{\sqrt{x(1-x)}}{n}\right)\\ &\le c\cdot\left(\sqrt{N_0}+1\right)^2\cdot\omega_2\left(f;\frac{\sqrt{x(1-x)}}{n}\right).\end{aligned}$$

For $n > N_0$, we proceed as follows. Define $i_n := [n\,\delta]$; then $1 \le i_n \le n/2$. Furthermore, let $\Delta_n$ be a sequence of partitions of the interval $[0,1]$ of the form

$$\Delta_n : 0 = x_{0,n} < x_{1,n} < \cdots < x_{n,n} = 1,$$

with

$$x_{k,n} = \begin{cases} \delta\cdot\sin^2\dfrac{k\pi}{2i_n}, & k = 0,\dots,i_n,\\ \delta + \dfrac{k-i_n}{n-2i_n}(1-2\delta), & k = i_n+1,\dots,n-i_n,\\ 1-\delta+\delta\sin^2\dfrac{(k-n+i_n)\pi}{2i_n}, & k = n-i_n+1,\dots,n.\end{cases}$$

In the sequel, we will write $i$ instead of $i_n$.

Notice that $x_{k,n}$ can be written as $x_{k,n} = \sin^2\theta_{k,n}$, $\theta_{k,n} = \arcsin\sqrt{x_{k,n}}$, $k = 0,\dots,n$. We will show that the requirements of Theorem 12 are satisfied.

Obviously, $\theta_{0,n} = 0$, $\theta_{n,n} = \pi/2$. Applying Lagrange's theorem for the function arcsin on the interval $[\sqrt{x_{k,n}}, \sqrt{x_{k+1,n}}]$, we obtain

$$\theta_{k+1,n} - \theta_{k,n} = \left(\sqrt{x_{k+1,n}} - \sqrt{x_{k,n}}\right)\cdot\frac{1}{\sqrt{1-\alpha_k^2}}, \qquad \alpha_k \in \left(\sqrt{x_{k,n}}, \sqrt{x_{k+1,n}}\right).$$

We distinguish among the following cases:

Case 1: $k = 0,\dots,i-1$,
Case 2: $k = i$,
Case 3: $k = i+1,\dots,n-i-1$,
Case 4: $k = n-i$,
Case 5: $k = n-i+1,\dots,n-2$,
Case 6: $k = n-1$.

CASE 1. We have

$$\sqrt{x_{k+1,n}} - \sqrt{x_{k,n}} = \sqrt{\delta}\cdot\sin\frac{\pi}{4i}\cdot\cos\frac{(2k+1)\pi}{4i} \le \sqrt{\delta}\cdot\frac{\pi}{4i} \quad\text{and}\quad \alpha_k < \sqrt{\delta}.$$

Since $(n/i)_{n>N_0}$ is a bounded sequence, it follows that there exists a constant $c_1$ such that

$$\theta_{k+1,n} - \theta_{k,n} \le \frac{c_1}{n}.$$

CASE 2. In this case, one has

$$\sqrt{x_{i+1,n}} - \sqrt{x_{i,n}} = \sqrt{\delta + \frac{1}{n-2i}(1-2\delta)} - \sqrt{\delta} \le \frac{1-2\delta}{2\sqrt{\delta}}\cdot\frac{1}{n-2i} \quad\text{and}\quad \alpha_i < \sqrt{1-\delta}.$$

Thus, we obtain

$$\theta_{i+1,n} - \theta_{i,n} \le \frac{1-2\delta}{2\sqrt{\delta}} \cdot \frac{1}{n-2i} \cdot \frac{1}{\sqrt{\delta}}.$$

Since the sequence $n/(n-2i)_{n>N_0}$ is bounded, it follows that there exists a constant $c_2$ such that

$$\theta_{i+1,n} - \theta_{i,n} \le \frac{c_2}{n}.$$

CASE 3. We have

$$\sqrt{x_{k+1,n}} - \sqrt{x_{k,n}} \le \frac{1}{n-2i} \cdot \frac{1-2\delta}{2\sqrt{\delta}} \quad \text{and} \quad \alpha_k < \sqrt{1-\delta}.$$

As in Case 2, there exists a constant $c_3$ such that

$$\theta_{k+1,n} - \theta_{k,n} \le \frac{c_3}{n}.$$

CASE 4.

$$\sqrt{x_{n-i+1,n}} - \sqrt{x_{n-i,n}} = \sqrt{1-\delta+\delta\,\sin^2\frac{\pi}{2i}} - \sqrt{1-\delta} \le \frac{\delta\,\sin^2(\pi/2i)}{2\sqrt{1-\delta}}.$$

It follows

$$\theta_{n-i+1,n} - \theta_{n-i,n} \le \frac{\delta\,\sin^2(\pi/2i)}{2\sqrt{1-\delta}} \cdot \frac{1}{\sqrt{\delta}\,\cos^2(\pi/2i)}.$$

Thus, there exists a constant $c_4$ such that

$$\theta_{n-i+1,n} - \theta_{n-i,n} \le \frac{c_4}{n}.$$

CASE 5.

$$\begin{aligned}\sqrt{x_{k+1,n}} - \sqrt{x_{k,n}} &\le \frac{\delta}{2\sqrt{1-\delta}} \cdot \left(\sin^2\frac{(k+1-n+i)\pi}{2i} - \sin^2\frac{(k-n+i)\pi}{2i}\right)\\ &= \frac{\delta}{2\sqrt{1-\delta}} \cdot \sin\frac{\pi}{2i} \cdot \sin\frac{(2k-2n+2i+1)\pi}{2i}.\end{aligned}$$

It follows that

$$\theta_{k+1,n} - \theta_{k,n} \le \frac{\delta}{2\sqrt{1-\delta}} \cdot \sin\frac{\pi}{2i} \cdot \sin\frac{(2n-2k-1)\pi}{2i} \cdot \frac{1}{\sqrt{\delta}\,\cos^2((k+1-n+i)\pi/2i)},$$

implying

$$\theta_{k+1,n} - \theta_{k,n} \le \frac{\delta}{2\sqrt{1-\delta}} \cdot \sin\frac{\pi}{2i} \cdot \sin\frac{(2n-2k-1)\pi}{2i} \cdot \frac{1}{\sqrt{\delta}\,\sin((n-k-1)\pi/2i)}.$$

Due to the choice of $k$, we have

$$\sin\frac{(2n-2k-1)\pi}{2i} \cdot \frac{1}{\sin((n-k-1)\pi/2i)} \le \frac{3\pi}{2}.$$

The previous two inequalities imply

$$\theta_{k+1,n} - \theta_{k,n} \le \frac{\sqrt{\delta}}{2\sqrt{1-\delta}} \cdot \frac{\pi}{2i} \cdot \frac{3\pi}{2}.$$

Since $(n/i)_{n>N_0}$ is bounded, it follows once more that there exists an absolute constant $c_5$ such that

$$\theta_{k+1,n} - \theta_{k,n} \leq \frac{c_5}{n}.$$

CASE 6. In this case, we have

$$x_{n,n} - x_{n-1,n} = \delta \cdot \sin^2 \frac{\pi}{2i} \leq \delta \cdot \frac{\pi^2}{4i^2}.$$

On the other hand,

$$x_{n,n} - x_{n-1,n} = 1 - \sin^2 \theta_{n-1,n} = \cos^2 \theta_{n-1,n} = \sin^2 \left(\frac{\pi}{2} - \theta_{n-1,n}\right).$$

But $\sin^2(\pi/2 - \theta_{n-1,n}) \geq 4/\pi^2 \cdot (\pi/2 - \theta_{n-1,n})^2$. It thus follows that $(\pi/2 - \theta_{n-1,n})^2 \leq \delta \cdot \pi^4/16i^2$, hence, $\theta_{n,n} - \theta_{n-1,n} \leq \sqrt{\delta} \cdot \pi^2/4i$. Again, using the fact that $(n/i)_{n>N_0}$ is bounded, it follows that there exists an absolute constant $c_6$ such that

$$\theta_{n,n} - \theta_{n-1,n} \leq \frac{c_6}{n}.$$

Writing now $c := \max_{i=1,\ldots,6} c_i$, we obtain

$$\theta_{k+1,n} - \theta_{k,n} \leq \frac{c}{n}, \qquad k = 0, \ldots, n-1.$$

Recall what we did: we showed that for all $n > N_0$ (a number related to $\delta$), there is a constant $c$ independent of $n$ (and $k$) such that for all the cases in question, we have

$$\theta_{k+1,n} - \theta_{k,n} \leq \frac{c}{n}.$$

The sequence of partitions $\Delta_n$, $n > N_0$, thus satisfies the assumptions given in Theorem 12, so that the proof of Theorem 14 is complete. ∎

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